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Decomposition's Dantzig–Wolfe applied to fuzzy multicommodity flow problems

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ABSTRACT

We present, in this paper, a method for solving linear programming problems with fuzzy costs based on the classical method of decomposition's Dantzig–Wolfe. Methods using decomposition techniques address problems that have a special structure in the set of constraints. An example of such a problem that has this structure is the fuzzy multicommodity flow problem. This problem can be modeled by a graph whose nodes represent points of supply, demand and passage of commodities, which travel on the arcs of the network. The objective is to determine the flow of each commodity on the arcs, in order to meet demand at minimal cost while respecting the capacity constraints of the arcs and the flow conservation constraints of the nodes. Using the theory of fuzzy sets, the proposed method aims to find the optimal solution, working with the problem in the fuzzy form during the resolution procedure.

1. Introduction

The initial work addressing the problem of multicommodity flow was proposed by Ford and Fulkerson [3], and Hu [7] in the early 1960s. The multicommodity flow problems have received much attention due to applicability in solving practical problems in diverse areas such as transport, telecommunications, among others. This problem can be modeled by a graph whose nodes represent points of supply, demand and passage of commodities, which travel on the arcs of the network. The objective is to determine the flow of each commodity on the arcs in order to meet demand at minimal cost while respecting the capacity constraints of the arcs, which limit the quantity of commodities that travel on it, and the flow conservation constraints of the nodes, which manage the flow at points of supply, demand and passage of commodities.

When modeling a real problem using the graph structure, as the information is not always accurate, one can have uncertainty both in the structure (nodes, edges) and in the parameters (cost, capacity, supply-demand). This problem can be studied and solved through the theory of fuzzy sets [2,9,12,16,17] which permits to treat mathematically certain levels of uncertainty. For Kaufmann and Gupta [8], the fuzzy set theory gave a form of mathematical precision to human cognitive processes that in many ways are imprecise and ambiguous by the standards of classical mathematics.

The fuzzy theory was formalized by Zadeh in 1965 [15]. According to him, it was possible to make progress in science from an imprecise environment. In 1973, based on the fuzzy theory, Zadeh formalized the fuzzy logic used in the fuzzy systems. He received the IEEE Medal of Honor in 1995 for his work in this area.

This paper presents a proposal for solving linear programming problems with fuzzy costs based on the classical method of decomposition's Dantzig–Wolfe [1], using the theory of fuzzy sets. Methods using decomposition techniques treat large problems that have a special structure in the set of constraints. An example of such a problem that has this structure is the fuzzy multicommodity flow problem.

There are few studies in the literature that deal with the fuzzy multicommodity flow problem. Two algorithms were proposed in the work of Ghatee and Hashemi [4]. The first, presenting uncertainty in cost, uses fuzzy shortest paths [10] to generate preferred paths and, then, a classical multicommodity flow problem (crisp) is used to determine the flow in these paths. The second, presenting uncertainties in cost, capacity and supply-demand, employs an order in trapezoidal numbers to transform the fuzzy problem in four classic multicommodity flow problems. The work of Verga et al. [14] presents an algorithm based on the Hernandes algorithm [5,6], which solves the minimal cost single-commodity flow problem with uncertainties in cost and capacity. The Okada and Soper's concept of dominant paths [10] is used to construct a representative subset of the set solutions for the fuzzy

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shortest path problem, called set of minimal non-dominant paths. With the theory of possibility [11], it was assigned to each path a degree of possibility to be minimal, thus ordering all paths. The flow is done through ordered paths.

Using the theory of fuzzy sets, the proposed method in the present study aims to find the optimal solution, working with the problem in the fuzzy form during the resolution procedure.

In the next section, some concepts of the theory of fuzzy sets used throughout this work are presented. In Section 3, the optimal solution of a linear programming problem with fuzzy costs is defined. In Section 4, the fuzzy multicommodity flow problem is presented. In Section 5, a method to solve linear programming problems with fuzzy costs based on the classical method of Dantzig–Wolfe decomposition is proposed. In Section 6, the numerical experiments in fuzzy multicommodity flow problems are presented, and in the last section, the conclusions and future work are described.

2. Concepts of the fuzzy theory

In the next section, some concepts of the theory of fuzzy sets used throughout this work are presented. Further details can be found in [2,9,12,16,17].

Any set classical can be characterized by a function, known as a characteristic function, whose definition is given below.

Definition 2.1. A characteristic function of a set A, defined in an universe $\mathbf{X}$,

$$f_A : \mathbf{X} \to \{0,1\}$$

is given by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The fundamental idea of fuzzy set is the gradual membership, i.e., relax this requirement assuming intermediate values between 0 and 1 to quantify the degree to which each element of the universe is associated with a class. The closer the value is to 1, the more compatible the element is with the properties that distinguish the class.

Definition 2.2. A fuzzy set A is described by a membership function that maps the elements of a universe $\mathbf{X}$ in the unit interval [0,1]:

$$\xi_A : \ \mathbf{X} \to [0,1].$$

A fuzzy set can be seen as a set of ordered pairs $\{x, \xi_A(x)\}$, of which x is an element of $\mathbf{X}$ and $\xi_A(x)$ denotes the degree of membership of x in A.

Definition 2.3. A triangular fuzzy number, denoted by $\tilde{a} = (m,\alpha,\beta)$, is described by the following membership function:

$$\xi_{\tilde{a}}(x) = \begin{cases} \dfrac{x-(m-\alpha)}{\alpha}, & m-\alpha < x < m, \\ 1, & x = m, \\ \dfrac{(m+\beta)-x}{\beta}, & m < x < m+\beta, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where m is the modal value, α is the left spread and β is the right spread. The values $m-\alpha$ and $m+\beta$ are the bounds, lower and upper, respectively.

Definition 2.4. A trapezoidal fuzzy number or a fuzzy interval, denoted by $\tilde{a} = (m_1,m_2,\alpha,\beta)$, is described by the following pertinence function:

$$\xi_{\tilde{a}}(x) = \begin{cases} \dfrac{x-(m_1-\alpha)}{\alpha}, & m_1-\alpha < x < m_1, \\ 1, & m_1 \le x \le m_2, \\ \dfrac{(m_2+\beta)-x}{\beta}, & m_2 < x < m_2+\beta, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where m_1 is the lower end of the modal interval, m_2 the upper end of the modal interval, α the left spread and β the right spread. The values $m_1-\alpha$ and $m_2+\beta$ are the bounds, lower and upper, respectively.

Definition 2.5. Let $\tilde{a}$ and $\tilde{b}$ be triangular fuzzy numbers, denoted by $\tilde{a} = (m_1,\alpha_1,\beta_1)$ and $\tilde{b} = (m_2,\alpha_2,\beta_2)$, and $k \in \mathbb{R}$. Operations are defined as:

(i) Sum:

$$\tilde{a}+\tilde{b} = (m_1+m_2, \alpha_1+\alpha_2, \beta_1+\beta_2).$$

(ii) Scalar product:

$$k\tilde{a} = (km_1, k\alpha_1, k\beta_1), \quad \text{if } k \ge 0,$$
$$k\tilde{a} = (km_1, -k\beta_1, -k\alpha_1), \quad \text{if } k < 0.$$

(iii) Subtraction:

$$\tilde{a}-\tilde{b} = \tilde{a}+(-\tilde{b}) = (m_1-m_2, \alpha_1+\beta_2, \beta_1+\alpha_2).$$

Definition 2.6. A triangular fuzzy vector of dimension k is given by

$$\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_k),$$

whose coordinates $\tilde{c}_1, \ldots, \tilde{c}_k$ are triangular fuzzy numbers.

Definition 2.7. Let $\tilde{c}$ be a triangular fuzzy vector of dimension k and $x = (x_1, \ldots, x_k)$ a vector in $\mathbb{R}^k$, the product of $\tilde{c}$ for x is the fuzzy number given by

$$\tilde{c}x = \sum_{i=1}^{k} \tilde{c}_i x_i.$$

Thus, if $A = [a_{ij}]$ is a real matrix $k \times l$, then the product of $\tilde{c}$ for A is the fuzzy vector of dimension l given by

$$\tilde{c}A = \left(\sum_{i=1}^{k} \tilde{c}_i a_{i1}, \sum_{i=1}^{k} \tilde{c}_i a_{i2}, \ldots, \sum_{i=1}^{k} \tilde{c}_i a_{il} \right).$$

There are many ways to compare fuzzy numbers, for example, dominance, possibility index, function ranking, etc. In this paper, two criteria are used to define if a triangular fuzzy number is less or more than another one.

Definition 2.8. The following criteria were defined by Kaufmann and Gupta [8].

1. First criterion: Let $\tilde{a} = (m,\alpha,\beta)$. The real number that "represents" the fuzzy number $\tilde{a}$ is given by

   $$\breve{a} = m + \tfrac{1}{4}(\beta-\alpha).$$

   It is said that $\tilde{a}$ is less than $\tilde{b}$, and is denoted by $\tilde{a} \underset{f}{<} \tilde{b}$, when $\breve{a} < \breve{b}$.
2. Second criterion: In this criterion, the modal value is used . Let $\tilde{a} = (m_1,\alpha_1,\beta_1)$ and $\tilde{b} = (m_2,\alpha_2,\beta_2)$ be such that $\breve{a} = \breve{b}$. It is said that $\tilde{a} \underset{f}{<} \tilde{b}$ when $m_1 < m_2$.

Similarly, it is defined $\tilde{a} \underset{f}{<} \tilde{b}$.

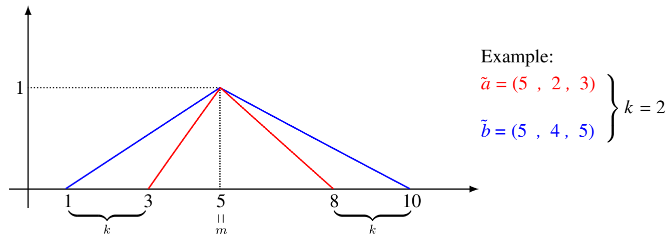

Fig. 1. Two triangular fuzzy numbers.

The first criterion of comparison is linear, i.e., if $\tilde{a}=\tilde{b}+k\tilde{c}$, then $\breve{a}=\breve{b}+k\breve{c}$, $k\in\mathbb{R}$. This criterion can be viewed as a linear ranking function $\Re : F(\mathbb{R})\rightarrow\mathbb{R}$ given by

$$\Re(\tilde{a})=m+\tfrac{1}{4}(\beta-\alpha),$$

where $\tilde{a}=(m,\alpha,\beta)$ and $F(\mathbb{R})$ denote the set of triangular fuzzy numbers.

When two fuzzy numbers have the same modal value and the spreads are different, but the differences for both right and left are equal, i.e., $\alpha_2=\alpha_1+k$ and $\beta_2=\beta_1+k$, $k\in\mathbb{R}$, they are considered equal as defined below (Fig. 1).

Definition 2.9. Let $\tilde{a}=(m_1,\alpha_1,\beta_1)$ and $\tilde{b}=(m_2,\alpha_2,\beta_2)$ be such that $\breve{a}=\breve{b}$ and $m_1=m_2$. In this case, it is said that $\tilde{a}$ is identical to $\tilde{b}$ and it is denoted by $\tilde{a}\underset{f}{=}\tilde{b}$.

Consider the fuzzy number $\tilde{0}=(0,0,0)$. By the above definition, $\tilde{a}\underset{f}{=}\tilde{0}$ if, and only if, $\tilde{a}=(0,\alpha,\alpha)$. In fact, if the modal value is zero, then $\breve{a}=0$ if, and only if, the left and right spreads are identical.

3. Fuzzy linear programming problem

Consider a linear programming problem whose exact values of the coefficients of the objective function are not known due to inaccurate information. Using the theory of fuzzy sets, this problem can be modeled as follows.

$$\begin{aligned}\min\quad & \tilde{z}=\tilde{c}\mathrm{x}\\ \text{s.t.}\quad & A\mathrm{x}=\mathrm{b},\\ & \mathrm{x}\geq 0,\end{aligned}\tag{3}$$

where A is a real matrix $m\times n$, such that $m<n$ and $rank(A)=m$, and $\tilde{c}$ is a triangular fuzzy vector of dimension n.

Through the criterion of comparison defined in Section 2, we extended the concept of optimal solution of a classical linear programming problem for a fuzzy linear problem.

Definition 3.1. A feasible solution x^* of problem (3) is optimal if, and only if, for every feasible solution x follows that $\tilde{c}\mathrm{x}^*\underset{f}{\leq}\tilde{c}\mathrm{x}$.

Consider the linear system $A\mathrm{x}=\mathrm{b}$. As $rank(A)=m$, there is a set of m linearly independent columns in A. Let B be the matrix formed by these columns and N the matrix formed by the remaining columns.

Thus, $A=[B\ N]$, where B, $m\times m$, is nonsingular. Decomposing x the same way, follows that $\mathrm{x}=[\mathrm{x}_B\ \mathrm{x}_N]'$ and $B\mathrm{x}_B+N\mathrm{x}_N=\mathrm{b}$. Therefore, $\mathrm{x}_B=B^{-1}\mathrm{b}-B^{-1}N\mathrm{x}_N$.

The solution $\mathrm{x}_B=B^{-1}\mathrm{b}$, $\mathrm{x}_N=0$ of the system $A\mathrm{x}=\mathrm{b}$ is called basic solution with respect to the base B. Decomposing $\tilde{c}$ in the same way, follows that $\tilde{c}=[\tilde{c}_B\ \tilde{c}_N]$, and if $B^{-1}\mathrm{b}\geq 0$, then the basic solution is feasible with the objective value $\tilde{z}=\tilde{c}_B\mathrm{x}_B$.

Writing $\tilde{z}=\tilde{c}\mathrm{x}=\tilde{c}_B\mathrm{x}_B+\tilde{c}_N\mathrm{x}_N$ in terms of nonbasic variables, follows that

$$\tilde{z}=\tilde{c}_B(B^{-1}\mathrm{b}-B^{-1}N\mathrm{x}_N)+\tilde{c}_N\mathrm{x}_N,$$

fixing x_j and making $x_k=0$, for $k\neq j$, in x_N,

$$\tilde{z}=\tilde{c}_B(B^{-1}\mathrm{b}-B^{-1}N_jx_j)+\tilde{c}_jx_j,$$

$$\tilde{z}\underset{f}{=}\tilde{c}_BB^{-1}\mathrm{b}-\tilde{c}_BB^{-1}N_jx_j+\tilde{c}_jx_j,$$

$$\tilde{z}\underset{f}{=}\tilde{c}_BB^{-1}\mathrm{b}+(\tilde{c}_j-\tilde{c}_BB^{-1}N_j)x_j.$$

Corresponding to each nonbasic variable x_j, the fuzzy number $\tilde{c}_BB^{-1}N_j$ is denoted by $\tilde{z}_j$. The following equation can be used to analyze the objective function $\tilde{z}$ if a nonbasic variable x_j enters the base.

$$\tilde{z}\underset{f}{=}\tilde{z}_0-(\tilde{z}_j-\tilde{c}_j)x_j,\tag{4}$$

$\tilde{z}_0$ is the objective value corresponding to the current basic feasible solution and $\tilde{z}_j-\tilde{c}_j$ is known as the relative cost of making the nonbasic variable x_j in the basic variable.

Theorem 3.1. *Consider Eq. (4). For $x_j>0$, follows that:*

1. *If $\tilde{z}_j-\tilde{c}_j\underset{f}{<}\tilde{0}$, then $\tilde{z}\underset{f}{>}\tilde{z}_0$.*
2. *If $\tilde{z}_j-\tilde{c}_j\underset{f}{>}\tilde{0}$, then $\tilde{z}\underset{f}{<}\tilde{z}_0$.*
3. *If $\tilde{z}_j-\tilde{c}_j\underset{f}{=}\tilde{0}$, then $\tilde{z}\underset{f}{=}\tilde{z}_0$.*

So, if $\tilde{z}_j-\tilde{c}_j\underset{f}{\leq}\tilde{0}$ for all nonbasic variables, then the current solution is optimal.

Demonstration. Consider $\tilde{z}_0=(m_1,\alpha_1,\beta_1)$, $\tilde{r}=\tilde{z}_j-\tilde{c}_j=(m_2,\alpha_2,\beta_2)$ and $k=x_j$. From Eq. (4), follows that

$$\tilde{z}\underset{f}{=}\tilde{z}_0-\tilde{r}k=(m_1,\alpha_1,\beta_1)-(km_2,k\alpha_2,k\beta_2)=(m_1-km_2,\alpha_1+k\beta_2,\beta_1+k\alpha_2).$$

Then

$$\breve{z}=m_1-km_2+\tfrac{1}{4}(\beta_1+k\alpha_2-\alpha_1-k\beta_2),$$

$$\breve{z}=m_1-km_2+\tfrac{1}{4}((\beta_1-\alpha_1)-(\beta_2-\alpha_2)k),$$

$$\breve{z}=m_1+\tfrac{1}{4}(\beta_1-\alpha_1)-(m_2+\tfrac{1}{4}(\beta_2-\alpha_2))k,$$

$$\breve{z}=\breve{z}_0-\breve{r}k.$$

1. If $\tilde{r}=\tilde{z}_j-\tilde{c}_j\underset{f}{<}\tilde{0}$ means that $\breve{r}<0$ or it means that $\breve{r}=0$ and $m_2<0$.
   (a) If $\breve{r}<0$, then $\breve{z}=\breve{z}_0-\breve{r}k\Rightarrow\breve{z}>\breve{z}_0$, because $k>0$, thus $\tilde{z}\underset{f}{>}\tilde{z}_0$.
   (b) If $\breve{r}=0$, then $\breve{z}=\breve{z}_0-\breve{r}k\Rightarrow\breve{z}=\breve{z}_0$. As $m_2<0$ and $k>0$, follows that $km_2<0\Rightarrow m_1-km_2>m_1$, thus $\tilde{z}\underset{f}{>}\tilde{z}_0$.
2. If $\tilde{r}=\tilde{z}_j-\tilde{c}_j\underset{f}{>}\tilde{0}$ means that $\breve{r}>0$ or it means that $\breve{r}=0$ and $m_2>0$.
   (a) If $\breve{r}>0$, then $\breve{z}=\breve{z}_0-\breve{r}k\Rightarrow\breve{z}<\breve{z}_0$, because $k>0$, thus $\tilde{z}\underset{f}{<}\tilde{z}_0$.
   (b) If $\breve{r}=0$, then $\breve{z}=\breve{z}_0-\breve{r}k\Rightarrow\breve{z}=\breve{z}_0$. As $m_2,k>0$, follows that $km_2>0\Rightarrow m_1-km_2<m_1$, thus $\tilde{z}\underset{f}{<}\tilde{z}_0$.
3. If $\tilde{z}_j-\tilde{c}_j\underset{f}{=}\tilde{0}$ means that $m_2=0$ and $\alpha_2=\beta_2$, therefore
   $$\tilde{z}\underset{f}{=}\tilde{z}_0-\tilde{r}k=(m_1,\alpha_1,\beta_1)-(0,k\alpha_2,k\alpha_2)=(m_1,\alpha_1+k\alpha_2,\beta_1+k\alpha_2).$$
   So,
   $$\breve{z}=m_1+\tfrac{1}{4}(\beta_1+k\alpha_2-(\alpha_1+k\alpha_2))=m_1+\tfrac{1}{4}(\beta_1-\alpha_1)=\breve{z}_0,$$
   as $\tilde{z}$ and $\tilde{z}_0$ have the same modal value, $\tilde{z}\underset{f}{=}\tilde{z}_0$.

By Definition 3.1, if $\tilde{z}_j - \tilde{c}_j \underset{\sim}{\leq} \tilde{0}$ for all nonbasic variables, then the current solution is optimal. $\square$

4. Fuzzy multicommodity flow problem

Let $G=(\mathcal{N},\mathcal{A})$ be a graph, where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ the arc set. Each arc is denoted by (i,j) for $i,j \in \mathcal{N}$, and K is the total number of commodities. The fuzzy multicommodity flow problem is formulated as the following linear programming problem *fuzzy*:

$$\min \quad \tilde{z} = \tilde{c}x = \sum_{k=1}^{K} \sum_{(i,j) \in \mathcal{A}} \tilde{c}_{ij}^k x_{ij}^k$$

$$\text{s.t.} \quad \begin{cases} \sum_{j:(i,j) \in \mathcal{A}} x_{ij}^k - \sum_{j:(j,i) \in \mathcal{A}} x_{ji}^k = d_i^k, & \forall i \in \mathcal{N},\ k=1,\ldots,K, \\ \sum_{k=1}^{K} x_{ij}^k \underset{\sim}{\leq} \tilde{b}_{ij}, & \forall (i,j) \in \mathcal{A}, \end{cases} \tag{5}$$

where

- x_{ij}^k is the flow of the commodity k on the arc (i,j);
- $\tilde{c}_{ij}^k$ is the fuzzy unit cost of the commodity k to traverse the arc (i,j);
- $\tilde{b}_{ij}$ is the fuzzy capacity of the arc (i,j);
- the symbol $\underset{\sim}{\leq}$ represents the relation of fuzzy order;
- d_i^k is the supply or demand of the commodity k in the node i:
  - If $d_i^k > 0$, i is the generator node of the commodity k.
  - If $d_i^k < 0$, i is the consumer node of the commodity k.
  - If $d_i^k = 0$, i is the passing node of the commodity k.

Generally, the costs are triangular fuzzy numbers, written as (m,α,β), where m is the modal value, α is the left spread and β is the right spread, and the capacities are fuzzy intervals written as $(m_1-\alpha, m_1, m_2, m_2+\beta)$, where m_1 is the extreme inferior of the modal interval, m_2 is the extreme superior of the modal interval, the values $m_1-\alpha$ and $m_2+\beta$ are the limits, inferior and superior, respectively.

We thought of applying methods using techniques of decomposition due to the fact that a part of constraints set of the problem (5) has a block diagonal structure, where it can be decomposed into several sets, each involving a commodity. Considering the crisp capacity, the problem (5) can be written as

$$\begin{array}{lllllll} \min & \tilde{c}^1 x^1 & + & \tilde{c}^2 x^2 & + \cdots + & \tilde{c}^K x^K & & \\ \text{s.t.} & A^1 x^1 & + & A^2 x^2 & + \cdots + & A^K x^K & \leq & b, \\ & D^1 x^1 & & & & & = & d^1, \\ & & & D^2 x^2 & & & = & d^2, \\ & & & & \ddots & & & \vdots \\ & & & & & D^K x^K & = & d^K, \\ & x^1, & & x^2, & \ldots, & x^K & \geq & 0, \end{array} \tag{6}$$

where

$$A^1 = \cdots = A^K = I_a, \quad \tilde{c}^k = [\tilde{c}_{ij}^k], \quad b = [b_{ij}],$$

$$d^k = [d_i^k], \quad D^1 = \cdots = D^K = M,$$

I_a is the identity matrix $a \times a$, M is the incidence matrix of the graph, consisting of 0 and ± 1, $M \in \mathbb{R}^{(n-1) \times a}$, $rank(M) = n-1$, $d^k \in \mathbb{R}^{n-1}$, n is the number of nodes and a the number of arcs. Excluding the non-negativity, it would be $(a+(n-1)\cdot K)$ constraints and $(a \cdot K)$ variables. Thus, the matrix $A=[A^1 \cdots A^K]$ has a dimension $a \times (a \cdot K)$.

In this paper, we approach only uncertainties in costs. Each fuzzy vector $\tilde{c}^k$ has a dimension corresponding to the block D^k. The coordinates of $\tilde{c}^k$ are triangular fuzzy numbers, written as $\tilde{a} = (m,\alpha,\beta)$, where m is the modal value, α is the left spread and β is the right spread.

In Section 5, a method that solves the fuzzy multicommodity flow problem with uncertainties in costs based on the classical method of Dantzig–Wolfe decomposition is proposed.

5. Proposed algorithm

The proposed algorithm aims to find the optimal solution for any fuzzy linear programming problem that can be written in the form of problem (6).

The constraints $A^1x^1 + A^2x^2 + \cdots + A^Kx^K \leq b$ of the problem (6) are called coupled constraints.

In problem (6), considering the constraints that have a block diagonal structure and the non-negativity, the sets $X^1, \ldots, X^K$ are defined below, each one involving a subset of variables, which do not appear in any other set.

$$X^k \equiv \{x^k : \ D^k x^k = d^k,\ x^k \geq 0\},$$

for $k=1,\ldots,K$.

Each set X^k defines a polyhedral set. Thus, each point of X^k can be written as a convex combination of the extreme points plus a non-negative linear combination of the extreme directions (if any) of X^k:

$$\begin{aligned} x^k &= \sum_{j=1}^{t_k} \lambda_j^k x_j^k + \sum_{j=1}^{l_k} \mu_j^k v_j^k, \\ &\sum_{j=1}^{t_k} \lambda_j^k = 1, \\ &\lambda_j^k \geq 0, \quad j=1,\ldots,t_k, \\ &\mu_j^k \geq 0, \quad j=1,\ldots,l_k, \end{aligned}$$

where $x_j^k\, j=1,\ldots,t_k$ are the extreme points and $v_j^k\, j=1,\ldots,l_k$ the extreme directions of X^k. Replacing each x^k in problem (6) by the previous representation, the next reformulated problem is obtained. Due to non-negativity of the variables x_j^k, v_j^k, λ_j^k and μ_j^k, the passage sums do not change the fuzzy numbers in the sense that the modal values and the spreads are the same.

$$\min \quad \sum_{k=1}^{K} \sum_{j=1}^{t_k} (\tilde{c}^k x_j^k) \lambda_j^k + \sum_{k=1}^{K} \sum_{j=1}^{l_k} (\tilde{c}^k v_j^k) \mu_j^k$$

$$\text{s.t.} \quad \sum_{k=1}^{K} \sum_{j=1}^{t_k} (A^k x_j^k) \lambda_j^k + \sum_{k=1}^{K} \sum_{j=1}^{l_k} (A^k v_j^k) \mu_j^k \leq b, \tag{7}$$

$$\sum_{j=1}^{t_k} \lambda_j^k = 1, \quad k=1,\ldots,K, \tag{8}$$

$$\begin{aligned} &\lambda_j^k \geq 0,\ j=1,\ldots,t_k \quad k=1,\ldots,K, \\ &\mu_j^k \geq 0,\ j=1,\ldots,l_k, \quad k=1,\ldots,K. \end{aligned}$$

This problem is called the Master Problem and its variables are $\lambda_j^k\, j=1,\ldots,t_k, k=1,\ldots,K$ and $\mu_j^k\, j=1,\ldots,l_k, k=1,\ldots,K$.

The constraints (7) are called master constraints (coming from the coupled ones) and (8) are the convexity constraints. In the fuzzy multicommodity flow problem there would be a constraints of type (7).

To solve the master problem, we make an adaptation of the revised simplex method for the case where the cost is a fuzzy vector. For this, the concepts of Section 2 are used. The block diagonal structure of the subset of the constraints can be further explored.

It is supposed that a basic feasible solution of the master problem is known with a base B $(a+K)\times(a+K)$. The base B must contain at least one variable λ_j^k of each block k in order to form the vector $\mathrm{x}=[\mathrm{x}^1\cdots\mathrm{x}^K]'$ solution of the original problem. It is supposed, also, that B^{-1}, $\overline{\mathrm{b}}=B^{-1}\binom{\mathrm{b}}{\mathbf{1}}$, $(\tilde{\omega},\tilde{\alpha})=[\tilde{\omega}_1\cdots\tilde{\omega}_a\tilde{\alpha}^1\cdots\ \tilde{\alpha}^K]=\tilde{\hat{\mathrm{c}}}_B B^{-1}$ are known, where $\tilde{\omega}$ and $\tilde{\alpha}$ are called fuzzy dual variables that correspond to the constraints (7) and (8), $\tilde{\hat{\mathrm{c}}}_B$ is the cost vector for the basic variables ($\tilde{\hat{c}}_j^k=\tilde{\mathrm{c}}^k\mathrm{x}_j^k$ for λ_j^k and $\tilde{\hat{c}}_j^k=\tilde{\mathrm{c}}^k v_j^k$ for μ_j^k). It is formed from the tableau of the Master Problem:

Reverse base	RHS
$(\tilde{\omega},\tilde{\alpha})$	$\tilde{\hat{\mathrm{c}}}_B\overline{\mathrm{b}}$
B^{-1}	$\overline{\mathrm{b}}$

The revised simplex method proceeds to conclude that the current solution is optimal or the problem is unlimited, otherwise, to decide to increase one nonbasic variable. For this, the relative cost $\tilde{z}_j^k-\tilde{\hat{c}}_j^k$ from each nonbasic variable is analyzed.

For λ_j^k: $\tilde{z}_j^k-\tilde{\hat{c}}_j^k=(\tilde{\omega},\tilde{\alpha})\binom{A^k\mathrm{x}_j^k}{e_k}-\tilde{\mathrm{c}}^k\mathrm{x}_j^k=\tilde{\omega}A^k\mathrm{x}_j^k+\tilde{\alpha}^k-\tilde{\mathrm{c}}^k\mathrm{x}_j^k$.

For μ_j^k : $\tilde{z}_j^k-\tilde{\hat{c}}_j^k=(\tilde{\omega},\tilde{\alpha})\binom{A^k v_j^k}{\mathbf{0}}-\tilde{\mathrm{c}}^k v_j^k=\tilde{\omega}A^k v_j^k-\tilde{\mathrm{c}}^k v_j^k$.

By Theorem 3.1, the current solution is optimal if $\tilde{z}_j^k-\tilde{\hat{c}}_j^k\underset{f}{\le}\tilde{0}$ for each nonbasic variable λ_j^k,μ_j^k.

Naturally $\tilde{z}_j^k-\tilde{\hat{c}}_j^k\underset{f}{=}\tilde{0}$ for each basic variable λ_j^k,μ_j^k. In fact,

$$\tilde{z}_j^k=\tilde{\hat{\mathrm{c}}}_B\underbrace{B^{-1}B_j^k}_{e_j^k}=\tilde{\hat{c}}_j^k\Rightarrow\tilde{z}_j^k-\tilde{\hat{c}}_j^k=\tilde{\hat{c}}_j^k-\tilde{\hat{c}}_j^k\underset{f}{=}\tilde{0}.$$

Once $\tilde{z}_j^k-\tilde{\hat{c}}_j^k\underset{f}{=}\tilde{0}$ for the basic variables, we have that $\max_{\substack{1\le j\le t_k\\1\le j\le l_k}}\{\tilde{z}_j^k-\tilde{\hat{c}}_j^k\}\underset{f}{\ge}\tilde{0}$.

- If $\max_{\substack{1\le j\le t_k\\1\le j\le l_k}}\{\tilde{z}_j^k-\tilde{\hat{c}}_j^k\}\underset{f}{=}\tilde{0}$, then $\tilde{z}_j^k-\tilde{\hat{c}}_j^k\underset{f}{\le}\tilde{0}$ for each nonbasic variable.
- If $\max_{\substack{1\le j\le t_k\\1\le j\le l_k}}\{\tilde{z}_j^k-\tilde{\hat{c}}_j^k\}\underset{f}{>}\tilde{0}$, then the corresponding nonbasic variable, λ_j^k or μ_j^k, is a candidate to become a basic variable.

The special structure that allowed the definition of the sets $X^1,\ldots,X^K$ facilitates the resolution of the problem. One can easily verify if $\tilde{z}_j^k-\tilde{\hat{c}}_j^k\underset{f}{\le}\tilde{0}$ is satisfied or not, solving each next subproblem, for $k=1,\ldots,K$.

$$\begin{aligned}&\max\quad(\tilde{\omega}A^k-\tilde{\mathrm{c}}^k)\mathrm{x}^k+\tilde{\alpha}^k\\&\text{s.t.}\quad \mathrm{x}^k\in X^k.\end{aligned}$$

To solve each subproblem, that the revised simplex was implemented for the case where the cost is a fuzzy vector. Except for the non-negativity, they are equality constraints.

Resolution of the subproblem k:

1. If the subproblem k provides an unbounded optimal objective value, then an extreme direction v_j^k such that $(\tilde{\omega}A^k-\tilde{\mathrm{c}}^k)v_j^k\underset{f}{>}\tilde{0}$ is found, whose corresponding variable μ_j^k is a candidate to enter the master base. In this case, the column $\binom{A^k v_j^k}{\mathbf{0}}$ is updated by premultiplying by B^{-1}, obtaining y_{kj}. The column $\binom{\tilde{z}_j^k-\tilde{\hat{c}}_j^k}{y_{kj}}$ is inserted into the tableau of the Master Problem.
2. If the subproblem k provides a strictly positive objective value, as $\tilde{z}_j^k-\tilde{\hat{c}}_j^k\underset{f}{=}\tilde{0}$ for the basic variables, it is concluded that an extreme point x_j^k was found whose corresponding variable λ_j^k is a candidate to enter the master base. In this case, the column $\binom{A^k\mathrm{x}_j^k}{e_k}$ is updated by premultiplying by B^{-1}, obtaining y_{kj}. The column $\binom{\tilde{z}_j^k-\tilde{\hat{c}}_j^k}{y_{kj}}$ is inserted into the tableau of the Master Problem.

Pivot at y_{kj_r} where the index r is determined by

$$\frac{\overline{b}_r}{y_{kj_r}}=\min_{1\le i\le a+K}\left\{\frac{\overline{b}_i}{y_{kj_i}};y_{kj_i}>0\right\}.$$

The first line of the tableau is not updated by pivoting. After the pivoting, we found the new matrix B^{-1} and the new vector $\overline{\mathrm{b}}$, then, with $\tilde{\hat{\mathrm{c}}}_B$ corresponding to the new base, the fuzzy dual variables are updated by $(\tilde{\omega},\tilde{\alpha})=\tilde{\hat{\mathrm{c}}}_B B^{-1}$, and the objective value is updated by $\tilde{z}\underset{f}{=}\tilde{\hat{\mathrm{c}}}_B\overline{\mathrm{b}}$.

If none of the two previous cases occurs, then there is currently no candidate to enter the master base for the subproblem k.

If no subproblem provides a candidate to enter the base, then the optimal solution was found. On the other hand, one must select one of the candidates to enter the master base. One can use the rule of selecting the one with the largest $\tilde{z}_j^k-\tilde{\hat{c}}_j^k$ or the first, and so on.

The resolution of the subproblems provides a point x_j^k, which corresponds to an updated column $\binom{\tilde{z}_j^k-\tilde{\hat{c}}_j^k}{y_{kj}}$, that is why this procedure is known as "column generation scheme".

As the constraints of the master problem is of the inequality type, then, besides solving subproblems, it is necessary to check the relative costs for the slack variables s_i before the end:

$$\tilde{z}_{s_i}-\tilde{c}_{s_i}=(\tilde{\omega},\tilde{\alpha})\begin{pmatrix}e_i\\\mathbf{0}\end{pmatrix}-\tilde{0}=\tilde{\omega}_i.$$

In short, given a basic feasible solution of the master problem, by solving the subproblems it is possible to find an optimal solution of the original problem, due to the block diagonal structure.

5.1. Summary of the proposed algorithm

Considering the problem (6), the data entries are: $\tilde{\mathrm{c}}^k,A^k,\mathrm{b},D^k$ and d^k, $k=1,\ldots,K$, where K is the number of blocks. For the fuzzy multicommodity flow problem $A^1=\cdots=A^K=I_a$ and $D^1=\cdots=D^K=M$, the identity matrix and the graph of incidence matrix, respectively, K is the number of commodities and a is the number of arcs. It makes sense to denote the length of the vector b also for a.

Step 1. *Initialization*: Find an initial basic feasible solution of the master problem.

Step 2. *Master Step*:

2.1. For $k=1,\ldots,K$, solve the following subproblems:

$$\begin{aligned}&\max\quad(\tilde{\omega}A^k-\tilde{\mathrm{c}}^k)\mathrm{x}^k+\tilde{\alpha}^k\\&\text{s.t.}\quad \mathrm{x}^k\in X^k.\end{aligned}$$

Let x_j^k be an optical feasible basic solution and $\tilde{z}_j^k-\tilde{\hat{c}}_j^k$ be the objective value.

If $\tilde{z}_j^k-\tilde{\hat{c}}_j^k\underset{f}{=}\tilde{0}$ for $k=1,\ldots,K$ and if $\tilde{\omega}\underset{f}{\le}\tilde{\mathbf{0}}$ stop, the basic feasible solution of the last master step provides an optimal solution of the original problem. Otherwise, go to item 2.2.

2.2. If $\tilde{z}_j^k - \tilde{c}_j^k \underset{f}{>} \tilde{0}$ for some k go to 2.2.1, otherwise go to 2.2.2.

2.2.1. Select one of the extreme points x_j^k with objective value $\tilde{z}_j^k - \tilde{c}_j^k \underset{f}{>} \tilde{0}$.

Obtain $y_{kj} = B^{-1}\binom{A^k \mathrm{x}_j^k}{e_k}$ and insert the column $\binom{\tilde{z}_j^k - \tilde{c}_j^k}{y_{kj}}$ in the tableau of the Master Problem.

Pivot at y_{kj_r}, where the index r is determined by

$$\frac{\overline{b}_r}{y_{kj_r}} = \min_{1 \le i \le a+K} \left\{ \frac{\overline{b}_i}{y_{kj_i}}; y_{kj_i} > 0 \right\}.$$

Update the tableau and return to item 2.1.

2.2.2. The slack variable s_i is a candidate to enter the base.

Obtain $y_{s_i} = B^{-1}\binom{e_i}{\mathbf{0}}$ and insert the column $\binom{\tilde{\omega}_i}{y_{s_i}}$ in the tableau of the Master Problem. There is no need to calculate $y_{s_i} = B^{-1}\binom{e_i}{\mathbf{0}}$, because the result is the ith column of B^{-1}.

Pivot at $y_{s_{ir}}$, where the index r is determined by

$$\frac{\overline{b}_r}{y_{s_{ir}}} = \min_{1 \le q \le a+K} \left\{ \frac{\overline{b}_q}{y_{s_{iq}}}; y_{s_{iq}} > 0 \right\}.$$

Update the tableau and return to item 2.1.

Whenever a nondegenerate pivot is performed the master step provides an improved feasible solution of the original problem.

5.2. Numerical example

In this section, a simple problem of multicommodity flow with uncertainties in the costs is solved with didactic purpose, because it allows a detailed analysis.

As a solution the following are presented: the flow of each commodity on the arcs, the total flow on the arcs and the total cost (objective function $\tilde{z}$). The total cost is obtained by multiplying the cost by the flow of each commodity in each arc ($\tilde{z} = \tilde{c}\mathrm{x}$).

Given the network of three nodes and three arcs in Fig. 2, it was considered that two commodities, p_1 and p_2, have as origin the node 1 and as destination the node 3. Offers and demands of the commodities are: $d_1^1 = 5, d_1^2 = 6, d_3^1 = -5$ and $d_3^2 = -6$.

Thus,

$$A^1 = A^2 = I_3, \quad D^1 = D^2 = M = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix},$$

$$\mathrm{d}^1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{d}^2 = \begin{pmatrix} 6 \\ 0 \end{pmatrix},$$

where the first column of the incidence matrix M refers to the arc (1,2), the second to arc (2,3) and the third to arc (1,3).

Fig. 3 illustrates the polyhedral set $X^1 \in \mathbb{R}^3$ given by a line segment whose extreme points are $(5\ 5\ 0)'$ and $(0\ 0\ 5)'$. Similarly, the extreme points of X^2 are $(6\ 6\ 0)'$ and $(0\ 0\ 6)'$.

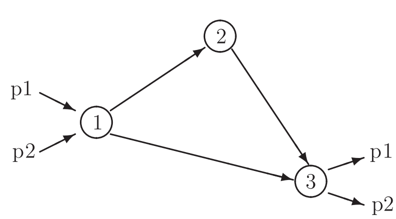

Fig. 2. Three nodes network.

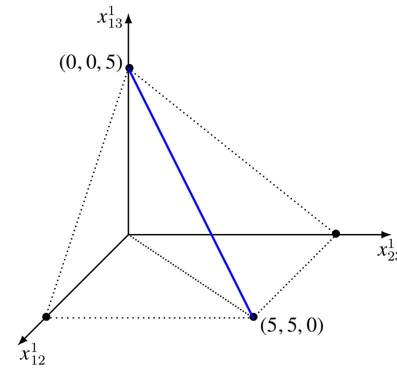

Fig. 3. Polyhedron X^1.

Table 1
Costs and capacities.

Arc	Cost of p_1	Cost of p_2	Capacity
1→2	(2,2,1)	(3,2,2)	5
2→3	(2,2,2)	(2,2,1)	6
1→3	(3,3,2)	(6,4,2)	7

The cost of each commodity and the capacity of each arc are given in Table 1.

Initialization step: There is an extreme point $\mathrm{x}_1^1 \in X^1$ and another $\mathrm{x}_1^2 \in X^2$ such that $A^1\mathrm{x}_1^1 + A^2\mathrm{x}_1^2 \le \mathrm{b}$, they are $\mathrm{x}_1^1 = (5\ 5\ 0)'$ and $\mathrm{x}_1^2 = (0\ 0\ 6)'$. Thus, considering the slack vector $\mathrm{s} = (s_1\ s_2\ s_3)'$, the initial tableau consists of:

	Reverse base			RHS
z	$\tilde{\mathbf{0}}$	$\tilde{c}^1\mathrm{x}_1^1$	$\tilde{c}^2\mathrm{x}_1^2$	$\tilde{c}^1\mathrm{x}_1^1 + \tilde{c}^2\mathrm{x}_1^2$
s	I_3	$-\mathrm{x}_1^1$	$-\mathrm{x}_1^2$	$\mathrm{b} - \mathrm{x}_1^1 - \mathrm{x}_1^2$
λ_1^1	$\mathbf{0}$	1	0	1
λ_1^2	$\mathbf{0}$	0	1	1

As $\mathrm{b} - \mathrm{x}_1^1 - \mathrm{x}_1^2 = (0\ 1\ 1)'$, $\tilde{c}^1\mathrm{x}_1^1 = (20, 20, 15)$, $\tilde{c}^2\mathrm{x}_1^2 = (36, 24, 12)$ and $\tilde{c}^1\mathrm{x}_1^1 + \tilde{c}^2\mathrm{x}_1^2 = (56, 44, 27)$, follows the initial tableau:

	Reverse base					RHS
z	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	(20,20,15)	(36,24,12)	(56,44,27)
s_1	1	0	0	−5	0	0
s_2	0	1	0	−5	0	1
s_3	0	0	1	0	−6	1
λ_1^1	0	0	0	1	0	1
λ_1^2	0	0	0	0	1	1

Master Step: First iteration:

1. Subproblem 1:

$$\begin{cases} \max & (\tilde{\omega}A^1 - \tilde{c}^1)\mathrm{x}^1 + \tilde{\alpha}^1 \\ \text{s.t.} & \mathrm{x}^1 \in X^1 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(2,2,1)x_{12}^1-(2,2,2)x_{23}^1-(3,3,2)x_{13}^1+(20,20,15) \\ \text{s.t.} & x_{12}^1+x_{13}^1=5, \\ & -x_{12}^1+x_{23}^1=0. \end{cases}$$

Solution: $\mathrm{x}_2^1=(0\ 0\ 5)'$ with objective value

$$\tilde{z}_2^1-\tilde{\tilde{c}}_2^1=-(15,15,10)+(20,20,15)=(5,30,30)\underset{f}{>}\tilde{0}.$$

2. Subproblem 2:

$$\begin{cases} \max & (\tilde{\omega}A^2-\tilde{c}^2)\mathrm{x}^2+\tilde{\alpha}^2 \\ \text{s.t.} & \mathrm{x}^2\in X^2 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(3,2,2)x_{12}^2-(2,2,1)x_{23}^2-(6,4,2)x_{13}^2+(36,24,12) \\ \text{s.t.} & x_{12}^2+x_{13}^2=6, \\ & -x_{12}^2+x_{23}^2=0. \end{cases}$$

Solution: $\mathrm{x}_2^2=(6\ 6\ 0)'$ with objective value

$$\tilde{z}_2^2-\tilde{\tilde{c}}_2^2=-(30,24,18)+(36,24,12)=(6,42,36)\underset{f}{>}\tilde{0}.$$

Adopting the rule of the more positive, as $(5,30,30)\underset{f}{>}(6,42,36)$, λ_2^1 is chosen to enter the master base. The column given by

$$\begin{pmatrix} \tilde{z}_2^1-\tilde{\tilde{c}}_2^1 \\ y_{12} \end{pmatrix}, \quad y_{12}=B^{-1}\begin{pmatrix} A^1\mathrm{x}_2^1 \\ e_1 \end{pmatrix}=\begin{pmatrix} -5 \\ -5 \\ 5 \\ 1 \\ 0 \end{pmatrix}$$

is inserted into the tableau:

	Reverse base					RHS	
z	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	(20,20,15)	(36,24,12)	(56,44,27)	(5,30,30)
s_1	1	0	0	−5	0	0	−5
s_2	0	1	0	−5	0	1	−5
s_3	0	0	1	0	−6	1	5
λ_1^1	0	0	0	1	0	1	1
λ_1^2	0	0	0	0	1	1	0

Pivot line: $r=3$. Therefore, the variable s_3 leaves the base. Pivoting to find the new matrix B^{-1} and the new vector $\overline{\mathrm{b}}$, and updating the first line:

$(\tilde{\omega},\tilde{\alpha})=\tilde{\tilde{c}}_BB^{-1}$ and $\tilde{z}\underset{f}{=}\tilde{\tilde{c}}_B\overline{\mathrm{b}}$

(getting the new $\tilde{\tilde{c}}_B$) we obtain the new tableau:

	Reverse base					RHS
z	$\tilde{0}$	$\tilde{0}$	(−1,6,6)	(20,20,15)	(42,60,48)	(55,43,26)
s_1	1	0	1	−5	−6	1
s_2	0	1	1	−5	−6	2
λ_2^1	0	0	1/5	0	−6/5	1/5
λ_1^1	0	0	−1/5	1	6/5	4/5
λ_1^2	0	0	0	0	1	1

Second iteration:

1. Subproblem 1:

$$\begin{cases} \max & (\tilde{\omega}A^1-\tilde{c}^1)\mathrm{x}^1+\tilde{\alpha}^1 \\ \text{s.t.} & \mathrm{x}^1\in X^1 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(2,2,1)x_{12}^1-(2,2,2)x_{23}^1-(4,9,8)x_{13}^1+(20,20,15) \\ \text{s.t.} & x_{12}^1+x_{13}^1=5, \\ & -x_{12}^1+x_{23}^1=0. \end{cases}$$

Solution: $\mathrm{x}_3^1=(0\ 0\ 5)'$ with objective value

$$\tilde{z}_3^1-\tilde{\tilde{c}}_3^1=-(20,45,40)+(20,20,15)=(0,60,60)\underset{f}{=}\tilde{0}.$$

2. Subproblem 2:

$$\begin{cases} \max & (\tilde{\omega}A^2-\tilde{c}^2)\mathrm{x}^2+\tilde{\alpha}^2 \\ \text{s.t.} & \mathrm{x}^2\in X^2 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(3,2,2)x_{12}^2-(2,2,1)x_{23}^2-(7,10,8)x_{13}^2+(42,60,48) \\ \text{s.t.} & x_{12}^2+x_{13}^2=6, \\ & -x_{12}^2+x_{23}^2=0. \end{cases}$$

Solution: $\mathrm{x}_3^2=(6\ 6\ 0)'$ with objective value

$$\tilde{z}_3^2-\tilde{\tilde{c}}_3^2=-(30,24,18)+(42,60,48)=(12,78,72)\underset{f}{>}\tilde{0}.$$

Only subproblem 2 provides a candidate this time, then λ_3^2 is chosen to enter the base master. The column given by

$$\begin{pmatrix} \tilde{z}_3^2-\tilde{\tilde{c}}_3^2 \\ y_{23} \end{pmatrix}, \quad y_{23}=B^{-1}\begin{pmatrix} A^2\mathrm{x}_3^2 \\ e_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ -6/5 \\ 6/5 \\ 1 \end{pmatrix}$$

is inserted into the tableau:

	Reverse base					RHS	
z	$\tilde{0}$	$\tilde{0}$	(−1,6,6)	(20,20,15)	(42,60,48)	(55,43,26)	(12,78,72)
s_1	1	0	1	−5	−6	1	0
s_2	0	1	1	−5	−6	2	0
λ_2^1	0	0	1/5	0	−6/5	1/5	−6/5
λ_1^1	0	0	−1/5	1	6/5	4/5	6/5
λ_1^2	0	0	0	0	1	1	1

Pivot line: $r=4$. Therefore, the variable λ_1^1 leaves. Pivoting and updating the first row, we obtain the new tableau:

	Reverse base					RHS
z	$\tilde{0}$	$\tilde{0}$	(1,7,6)	(10,45,45)	(30,24,18)	(47,39,26)
s_1	1	0	1	−5	−6	1
s_2	0	1	1	−5	−6	2
λ_2^1	0	0	0	1	0	1
λ_3^2	0	0	−1/6	5/6	1	2/3
λ_1^2	0	0	1/6	−5/6	0	1/3

Third iteration:

1. Subproblem 1:

$$\begin{cases} \max & (\tilde{\omega}A^1 - \tilde{c}^1)x^1 + \tilde{\alpha}^1 \\ \text{s.t.} & x^1 \in X^1 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(2,2,1)x^1_{12} - (2,2,2)x^1_{23} - (2,9,9)x^1_{13} + (10,45,45) \\ \text{s.t.} & x^1_{12} + x^1_{13} = 5, \\ & -x^1_{12} + x^1_{23} = 0. \end{cases}$$

Solution: $x^1_4 = (0\ 0\ 5)'$ with objective value

$\tilde{z}^1_4 - \tilde{\hat{c}}^1_4 = -(10,45,45) + (10,45,45) = (0,90,90) \underset{f}{=} \tilde{0}$.

2. Subproblem 2:

$$\begin{cases} \max & (\tilde{\omega}A^2 - \tilde{c}^2)x^2 + \tilde{\alpha}^2 \\ \text{s.t.} & x^2 \in X^2 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(3,2,2)x^2_{12} - (2,2,1)x^2_{23} - (5,10,9)x^2_{13} + (30,24,18) \\ \text{s.t.} & x^2_{12} + x^2_{13} = 6, \\ & -x^2_{12} + x^2_{23} = 0. \end{cases}$$

Solution: $x^2_4 = (6\ 6\ 0)'$ with objective value

$\tilde{z}^2_4 - \tilde{\hat{c}}^2_4 = -(30,24,18) + (30,24,18) = (0,42,42) \underset{f}{=} \tilde{0}$.

Subproblems do not provide candidates to enter the base master. Are there slack variable candidates to enter the base?

Noting that $\tilde{z}_{s_i} - \tilde{c}_{s_i} = \tilde{\omega}_i$, as $\tilde{\omega}_3 = (1,7,6) \underset{f}{>} \tilde{0}$, s_3 is a candidate to enter the master base. The column given by

$$\begin{pmatrix} \tilde{z}_{s_3} - \tilde{c}_{s_3} \\ y_{s_3} \end{pmatrix}, \quad y_{s_3} = B^{-1}\begin{pmatrix} e_3 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1/6 \\ 1/6 \end{pmatrix}$$

is inserted into the tableau:

	Reverse base					RHS	
z	$\tilde{0}$	$\tilde{0}$	(1,7,6)	(10,45,45)	(30,24,18)	(47,39,26)	(1,7,6)
s_1	1	0	1	−5	−6	1	1
s_2	0	1	1	−5	−6	2	1
λ^1_2	0	0	0	1	0	1	0
λ^2_3	0	0	−1/6	5/6	1	2/3	−1/6
λ^2_1	0	0	1/6	−5/6	0	1/3	−1/6

Pivot line: $r=1$. Therefore, the variable s_1 leaves. Pivoting and updating the first row, we obtain the new tableau:

	Reverse base					RHS
z	(−1,6,7)	$\tilde{0}$	$\tilde{0}$	(15,15,10)	(36,24,12)	(46,39,27)
s_3	1	0	1	−5	−6	1
s_2	−1	1	0	0	0	1
λ^1_2	0	0	0	1	0	1
λ^2_3	1/6	0	0	0	0	5/6
λ^2_1	−1/6	0	0	0	1	1/6

Fourth iteration:

1. Subproblem 1:

$$\begin{cases} \max & (\tilde{\omega}A^1 - \tilde{c}^1)x^1 + \tilde{\alpha}^1 \\ \text{s.t.} & x^1 \in X^1 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(3,9,7)x^1_{12} - (2,2,2)x^1_{23} - (3,3,2)x^1_{13} + (15,15,10) \\ \text{s.t.} & x^1_{12} + x^1_{13} = 5, \\ & -x^1_{12} + x^1_{23} = 0. \end{cases}$$

Solution: $x^1_5 = (0\ 0\ 5)'$ with objective value

$\tilde{z}^1_5 - \tilde{\hat{c}}^1_5 = -(15,15,10) + (15,15,10) = (0,25,25) \underset{f}{=} \tilde{0}$.

2. Subproblem 2:

$$\begin{cases} \max & (\tilde{\omega}A^2 - \tilde{c}^2)x^2 + \tilde{\alpha}^2 \\ \text{s.t.} & x^2 \in X^2 \end{cases} \Rightarrow$$

$$\begin{cases} \max & -(4,9,8)x^2_{12} - (2,2,1)x^2_{23} - (6,4,2)x^2_{13} + (36,24,12) \\ \text{s.t.} & x^2_{12} + x^2_{13} = 6, \\ & -x^2_{12} + x^2_{23} = 0. \end{cases}$$

Solution: $x^2_5 = (6\ 6\ 0)'$ with objective value

$\tilde{z}^2_5 - \tilde{\hat{c}}^2_5 = -(36,66,54) + (36,24,12) = (0,78,78) \underset{f}{=} \tilde{0}$.

Subproblems do not provide candidates to enter the base master. Are there slack variable candidates to enter the base?

As $\tilde{\omega} \underset{f}{\leq} \tilde{0}$, there are no slack variable candidates to enter the base. Thus, the basic feasible solution of the previous iteration provides an optimal solution of the original problem:

$$x^* = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

given by

$$x^1 = \lambda^1_2 x^1_2 = 1 \cdot (0\ 0\ 5)',$$

$$x^2 = \lambda^2_1 x^2_1 + \lambda^2_3 x^2_3 = 1/6 \cdot (0\ 0\ 6)' + 5/6 \cdot (6\ 6\ 0)' = (5\ 5\ 1)'.$$

Thus $x^* = (0\ 0\ 5\ 5\ 5\ 1)'$.

The flows of the commodities on each arc are given in Table 2.

The total cost is $\tilde{z}^* = (46,39,27)$.

Table 2
Flows of the commodities.

Arc	Flow of p_1	Flow of p_2	Total flow
(1,2)	0	5	5
(2,3)	0	5	5
(1,3)	5	1	6

6. Numerical experiments

The computational tests of the proposed algorithm in Section 5 are presented in this section. This algorithm was implemented in Matlab 7.0.1 and run on an Intel Core 2 Duo, 2.53 GHz and 3 GB of RAM. We used three fuzzy multicommodity flow problems to test the algorithm. The first problem is given by a small network with six nodes and nine arcs. The second is given by a network used in [4] containing 13 nodes and 31 arcs. The third is given by a real network, the COST 239 European Optical Network [13], with 11 nodes and arcs with double direction totaling 50 arcs.

As a solution the following are presented: the flow of each commodity on the arcs and the total cost (objective function $\tilde{z}$). The total cost is obtained by multiplying the cost by the flow of each commodity in each arc ($\tilde{z} = \tilde{c}x$). We also present the results of problems without uncertainties using the command `linprog` of the MatLab. The command `linprog` from the optimization toolbox solves linear programming problems (Syntax: `[x,fval]= linprog(f,A,b,Aeq,beq,lb,ub)`).

6.1. Problem 1

The first problem is given by the network of Fig. 4. We considered two commodities, p_1 and p_2, both have as source nodes 1 and 2, and as destination nodes 4, 5 and 6. The supply and demand are given in Table 3.

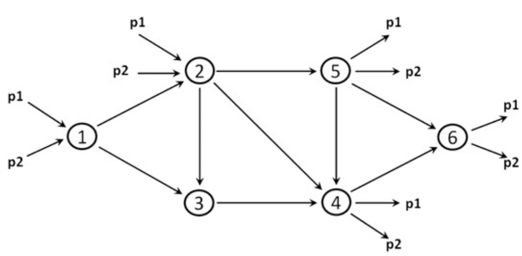

Fig. 4. Graph of Problem 1.

Table 3
Supply and demand—six nodes.

Node	Commodity p_1		Commodity p_2	
	Supply	Demand	Supply	Demand
1	1	0	1	0
2	2.5	0	2.5	0
4	0	1.5	0	1.5
5	0	1	0	1
6	0	1	0	1

Table 4
Costs and capacities.

Arc	(i,j)	Capacity	Cost of p_1	Cost of p_2
1	(1,2)	3	(2,1,1)	(3,1,1)
2	(1,3)	1.5	(3,2,1)	(5,3,1)
3	(2,3)	2	(3,3,2)	(5,5,3)
4	(2,4)	2	(6,4,2)	(9,6,3)
5	(2,5)	7	(5,1,1)	(8,3,3)
6	(3,4)	3	(5,2,1)	(7,5,3)
7	(4,6)	4	(2,1,1)	(3,1,1)
8	(5,4)	4	(5,4,2)	(8,6,3)
9	(5,6)	2	(3,1,1)	(5,2,2)

The cost of each commodity to traverse each arc and the capacity of each arc are given in Table 4.

1. Decomposition method: The algorithm made 11 iterations in 0.1570 s to get the optimal solution.

   The flows of the commodities on each arc are given in Table 5. The total cost was

   $\tilde{z} = (61.0, 29.5, 19.5)$.

   It is observed that the total flow of commodities on the arcs, 2 and 4, is equal to capacity of the respective arcs. In this case, if the capacities are increased, then the total cost to meet the same demand may decrease.
2. MatLab: To solve the problem without uncertainties, in other words, considering the costs as real numbers (crisp costs), we used MatLab (command `linprog`). Considering the modal value of the fuzzy cost, the flows of the commodities on each arc are given in Table 6. The total cost was 61.

Note: The total cost of crisp problem coincided with the modal value of the total cost of the fuzzy problem, which was $\tilde{z} = (61, 29.5, 19.5)$. However, the optimum flow of the crisp problem found by MatLab is different from the optimum flow of the fuzzy problem found by the decomposition method (Tables 5 and 6).

Considering that the fuzzy cost vector given in Table 4 and the optimum flow of the crisp problem given in Table 6, the total cost would be $\tilde{z}_M = (61.0000, 29.0034, 19.2517)$. Comparing the two fuzzy costs, $\tilde{z}_M$ and $\tilde{z}$, through the criterion of comparison defined in Section 2, it follows that

$\check{z}_M = 58.5621 \quad \text{and} \quad \check{z} = 58.5000$

so

$\tilde{z}_M \underset{f}{>} \tilde{z}$.

Note: The considered problem presents uncertainties only in costs, then an optimal solution of the crisp problem (crisp costs) is a feasible solution of the fuzzy problem. So, the solution found by

Table 5
Flows of the commodities—decomposition.

Arc	(i,j)	Capacity	Flow of p_1	Flow of p_2	Total flow
1	(1,2)	3	0.5	0	0.5
2	(1,3)	1.5	0.5	1.0	1.5
3	(2,3)	2	0	0	0
4	(2,4)	2	1.0	1.0	2.0
5	(2,5)	7	2.0	1.5	3.5
6	(3,4)	3	0.5	1.0	1.5
7	(4,6)	4	0	0.5	0.5
8	(5,4)	4	0	0	0
9	(5,6)	2	1.0	0.5	1.5

Table 6
Flows of the commodities—MatLab.

Arc	(i,j)	Capacity	Flow of p_1	Flow of p_2	Total flow
1	(1,2)	3	0.2517	0.2483	0.5000
2	(1,3)	1.5	0.7483	0.7517	1.5000
3	(2,3)	2	0	0	0
4	(2,4)	2	0.7517	1.2483	2.0000
5	(2,5)	7	2.0000	1.5000	3.5000
6	(3,4)	3	0.7483	0.7517	1.5000
7	(4,6)	4	0	0.5000	0.5000
8	(5,4)	4	0	0	0
9	(5,6)	2	1.0000	0.5000	1.5000

the proposed method must be better or equal to solution found by MatLab considering the crisp problem.

The data of the Problem 1 will be changed to show that the modal value of the fuzzy total cost does not always coincide with the crisp total cost. The new data to the Problem 1 are given in Tables 7 and 8.

On the arc (1,2), the fuzzy cost of the commodity 1 is greater than the fuzzy cost of the commodity 2:

$$\tilde{c}^1_{12} = (50, 10, 500) \Rightarrow \check{c}^1_{12} = 172.5,$$

$$\tilde{c}^2_{12} = (200, 200, 10) \Rightarrow \check{c}^2_{12} = 152.5,$$

but this is reversed if considered only as the modal value.

Solving the problem without uncertainty through the MatLab (command `linprog`), considering the modal value of the fuzzy costs, the total cost was 590 000. Solving the fuzzy problem through the decomposition method the total cost was $\tilde{z} = (615\,000, 143\,000, 156\,500)$. The optimal flows are given in Table 9.

The optimum flow of the crisp problem found by MatLab is different from the optimum flow of the fuzzy problem found by the decomposition method. It is observed that the modal value of the total cost obtained by the decomposition method ($10^3 \cdot 615$) is greater than the total cost obtained by MatLab ($10^3 \cdot 590$). Multiplying the flow found by MatLab by the fuzzy cost we obtain $\tilde{z}_M = 10^3 \cdot (590, 106{,}234.5)$. Through the criterion of comparison it follows that

$$\check{z}_M = 622\,125 \quad \text{and} \quad \check{z} = 618\,375$$

so

$$\tilde{z}_M \underset{f}{>} \tilde{z}.$$

The results obtained in the two problems show that the proposed decomposition method had a great performance, as it considers the uncertainties in the costs to find the optimal solutions, which are better than those found by MatLab considering the crisp problem.

Table 7
New data of the Problem 1.

Node	Commodity p_1		Commodity p_2	
	Supply	Demand	Supply	Demand
1	100	0	100	0
2	250	0	250	0
3	0	0	0	0
4	0	150	0	150
5	0	100	0	100
6	0	100	0	100

Table 8
Costs and capacities.

Arc	(i,j)	Capacity	Cost of p_1	Cost of p_2
1	(1,2)	300	(50,10,500)	(200,200,10)
2	(1,3)	200	(400,10,200)	(400,10,200)
3	(2,3)	200	(300,20,300)	(500,30,500)
4	(2,4)	200	(600,400,20)	(900,600,30)
5	(2,5)	700	(500,10,100)	(800,30,300)
6	(3,4)	300	(500,10,200)	(700,30,500)
7	(4,6)	400	(200,10,100)	(300,10,100)
8	(5,4)	400	(500,20,400)	(800,30,600)
9	(5,6)	200	(300,10,100)	(500,20,200)

Table 9
Flows of the commodities—MatLab and decomposition.

Arc	Capacity	MatLab			Decomposition		
		Flow of p_1	Flow of p_2	Total flow	Flow of p_1	Flow of p_2	Total flow
1	300	100	0	100	0	100	100
2	200	0	100	100	100	0	100
3	200	0	0	0	0	0	0
4	200	150	50	200	50	150	200
5	700	200	200	400	200	200	400
6	300	0	100	100	100	0	100
7	400	0	0	0	0	0	0
8	400	0	0	0	0	0	0
9	200	100	100	200	100	100	200

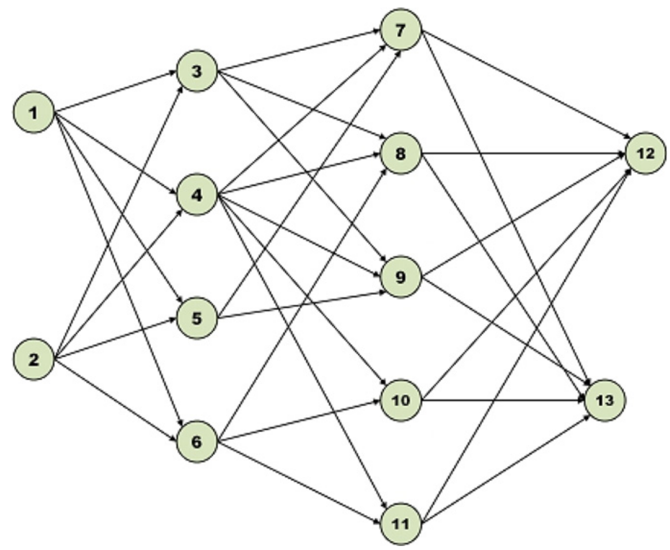

Fig. 5. Graph of Problem 2.

Table 10
Supplies and demands—13 nodes.

Node	Commodity p_1		Commodity p_2	
	Supply	Demand	Supply	Demand
1	100	0	100	0
2	80	0	100	0
3	0	0	0	0
4	70	0	70	0
5	0	0	0	0
6	0	0	0	0
7	0	20	0	40
8	0	20	0	10
9	0	20	0	30
10	0	20	0	20
11	0	0	0	10
12	0	30	0	70
13	0	140	0	90

6.2. Problem 2

The second problem is given by a network used in [4] containing 13 nodes and 31 arcs. Fig. 5 illustrates this network.

We considered two commodities, p_1 and p_2, with supplies and demands given in Table 10.

The costs of the commodities and the capacities of the arcs were taken from [4]. Trapezoidal fuzzy costs given in [4] have been adapted for triangular fuzzy numbers as follows:

$$\tilde{c} = (m_1-\alpha, m_1, m_2, m_2+\beta) \Rightarrow \tilde{c} = \left(m_1-\alpha, \frac{m_1+m_2}{2}, m_2+\beta\right),$$

where $m_1-\alpha$ is the lower bound, $(m_1+m_2)/2$ is the modal value and $m_2+\beta$ is the upper bound.

For example:

$$\tilde{c} = (20, 30, 37, 45) \Rightarrow \tilde{c} = (20, 33.5, 45).$$

Writing in the form (m,α,β), it follows that $\tilde{c} = (33.5, 13.5, 11.5)$.

The cost of p_2 is 90% of the cost of p_1. The capacities of the arcs and the fuzzy costs (modal value and spreads) of the commodities are given in Table 11, and the solution of the fuzzy problem found by decomposition method and the solution of the crisp problem found by MatLab (command `linprog`).

It can be observed in Table 11 that the optimal solution of the fuzzy problem found by the decomposition method is equal to the solution of the crisp problem. The decomposition method made 81 iterations in 9.3750 s. The total cost was

$$\tilde{z} = (54\,172.7, 9\,152.9, 10\,956.6).$$

Increasing spreads only in the first two arcs, highlighted in bold in Table 12, the decomposition method finds a different solution. The altered flows highlighted in bold in Table 12, it can be observed that in addition to changing the quantities of commodities have changed the paths traversed. For example, in node 1 the supply of the commodity 2 was distributed by the arcs 2, 3 and 4, now it is distributed by the arcs 1, 3 and 4. The total cost was

$$\tilde{z} = (54\,195.2, 13\,600.4, 17\,974.8).$$

Multiplying the flow found by MatLab by the fuzzy cost we obtain

$$\tilde{z}_M = (54\,172.7, 13\,262.8, 18\,456.5).$$

Comparing the two fuzzy costs, $\tilde{z}_M$ and $\tilde{z}$, through the criterion of comparison it follows that

$$\check{z}_M = 55\,471.2 \quad \text{and} \quad \check{z} = 55\,288.8$$

so

$$\tilde{z}_M \underset{f}{>} \tilde{z}.$$

The decomposition method found a better solution than that obtained by MatLab considering the crisp problem.

6.3. Problem 3

The third problem is given by a real network, the COST 239 European Optical Network [13], with 11 nodes and arcs of double-direction totaling 50 arcs. A arc of double-direction $(i \leftrightarrow j)$ represents a cycle of length two between the considered nodes. Fig. 6 illustrates this network.

In the telecommunications area, there are optimization problems in optical fiber networks, in which the commodities can be transfer of data, voice, image, etc.

We considered four commodities, $p_1, \ldots, p_4$, with supplies and demands given in Table 13.

Table 11
MatLab and decomposition.

Arc			Cost of p_1			Cost of p_2			Flow on arc			
No.	(i, j)	Capacity	Modal value	Left spread	Right spread	Modal value	Left spread	Right spread	Matlab		Decom.	
									p_1	p_2	p_1	p_2
1	(1,3)	75	33.5	13.5	11.5	30.2	12.2	10.4	75	–	75	–
2	(1,4)	45	34.0	4.0	8.0	30.6	3.6	7.2	6	39	6	39
3	(1,5)	93	76.5	10.5	12.5	68.9	9.5	11.3	–	33	–	33
4	(1,6)	47	35.5	7.5	9.5	32.0	6.8	8.6	19	28	19	28
5	(2,3)	42	58.0	11.0	5.0	52.2	9.9	4.5	–	42	–	42
6	(2,4)	85	18.0	12.0	7.0	16.2	10.8	6.3	80	5	80	5
7	(2,5)	53	106.5	7.5	5.5	95.9	6.8	5.0	–	33	–	33
8	(2,6)	20	59.0	1.0	6.0	53.1	0.9	5.4	–	20	–	20
9	(3,7)	67	53.0	11.5	10.5	48.2	10.4	9.5	41	16	41	16
10	(3,8)	84	60.0	8.0	7.0	54.0	7.2	6.3	32	26	32	26
11	(3,9)	2	39.0	6.0	8.0	35.1	5.4	7.2	2	–	2	–
12	(4,7)	68	54.5	11.5	9.5	49.1	10.4	8.6	9	19	9	19
13	(4,8)	38	34.0	11.0	6.0	30.6	9.9	5.4	38	–	38	–
14	(4,9)	83	62.0	4.0	18.0	55.8	3.6	16.2	18	65	18	65
15	(4,10)	50	80.5	4.5	7.5	72.5	4.1	6.8	20	30	20	30
16	(4,11)	71	60.5	7.5	7.5	54.5	6.8	6.8	71	–	71	-
17	(5,7)	43	75.0	11.0	9.0	67.5	9.9	8.1	–	36	-	36
18	(5,9)	30	31.5	10.5	7.5	28.4	9.5	6.8	–	30	–	30
19	(6,8)	19	43.0	5.0	9.0	38.7	4.5	8.1	19	–	19	–
20	(6,10)	19	88.0	10.0	11.0	79.2	9.0	9.9	–	19	–	19
21	(6,11)	68	72.5	4.5	8.5	65.3	4.1	7.7	–	29	–	29
22	(7,12)	30	51.5	5.5	10.5	46.4	5.0	9.5	30	–	30	-
23	(7,13)	54	67.5	10.5	4.5	60.8	9.5	4.1	–	31	–	31
24	(8,12)	15	80.0	1.0	11.0	72.0	0.9	9.9	–	15	–	15
25	(8,13)	70	10.5	4.5	6.5	9.5	4.1	5.9	69	1	69	1
26	(9,12)	38	64.0	4.0	5.0	57.6	3.6	4.5	–	26	–	26
27	(9,13)	86	11.5	6.5	13.5	10.4	5.9	12.2	–	39	–	39
28	(10,12)	85	52.5	10.5	11.5	47.3	9.5	10.4	–	29	–	29
29	(10,13)	59	36.0	6.0	4.0	32.4	5.4	3.6	–	–	–	–
30	(11,12)	50	91.5	4.5	2.5	82.4	4.1	2.3	–	–	–	–
31	(11,13)	90	6.5	4.5	13.5	5.9	4.1	12.2	71	19	71	19

Table 12
MatLab and decomposition—problem changed.

Arc			Cost of p_1			Cost of p_2			Flow on arc			
No.	(i,j)	Capacity	Modal value	Left spread	Right spread	Modal value	Left spread	Right spread	Matlab		Decom.	
									p_1	p_2	p_1	p_2
1	(1,3)	75	33.5	13.5	**111.5**	30.2	12.2	**100.4**	75	–	**24**	**51**
2	(1,4)	45	34.0	**104.0**	8.0	30.6	**93.6**	7.2	6	39	**45**	**0**
3	(1,5)	93	76.5	10.5	12.5	68.9	9.5	11.3	–	33	–	33
4	(1,6)	47	35.5	7.5	9.5	32.0	6.8	8.6	19	28	**31**	**16**
5	(2,3)	42	58.0	11.0	5.0	52.2	9.9	4.5	–	42	–	42
6	(2,4)	85	18.0	12.0	7.0	16.2	10.8	6.3	80	5	80	5
7	(2,5)	53	106.5	7.5	5.5	95.9	6.8	5.0	–	33	–	33
8	(2,6)	20	59.0	1.0	6.0	53.1	0.9	5.4	–	20	–	20
9	(3,7)	67	53.0	11.5	10.5	48.2	10.4	9.5	41	16	**22**	**35**
10	(3,8)	84	60.0	8.0	7.0	54.0	7.2	6.3	32	26	**0**	**58**
11	(3,9)	2	39.0	6.0	8.0	35.1	5.4	7.2	2	–	2	–
12	(4,7)	68	54.5	11.5	9.5	49.1	10.4	8.6	9	19	**28**	**0**
13	(4,8)	38	34.0	11.0	6.0	30.6	9.9	5.4	38	–	38	–
14	(4,9)	83	62.0	4.0	18.0	55.8	3.6	16.2	18	65	**38**	**45**
15	(4,10)	50	80.5	4.5	7.5	72.5	4.1	6.8	20	30	20	30
16	(4,11)	71	60.5	7.5	7.5	54.5	6.8	6.8	71	–	71	–
17	(5,7)	43	75.0	11.0	9.0	67.5	9.9	8.1	–	36	–	36
18	(5,9)	30	31.5	10.5	7.5	28.4	9.5	6.8	–	30	–	30
19	(6,8)	19	43.0	5.0	9.0	38.7	4.5	8.1	19	–	19	–
20	(6,10)	19	88.0	10.0	11.0	79.2	9.0	9.9	–	19	–	19
21	(6,11)	68	72.5	4.5	8.5	65.3	4.1	7.7	–	29	**12**	**17**
22	(7,12)	30	51.5	5.5	10.5	46.4	5.0	9.5	30	–	30	–
23	(7,13)	54	67.5	10.5	4.5	60.8	9.5	4.1	–	31	–	31
24	(8,12)	15	80.0	1.0	11.0	72.0	0.9	9.9	–	15	–	15
25	(8,13)	70	10.5	4.5	6.5	9.5	4.1	5.9	69	1	**37**	**33**
26	(9,12)	38	64.0	4.0	5.0	57.6	3.6	4.5	–	26	–	26
27	(9,13)	86	11.5	6.5	13.5	10.4	5.9	12.2	–	39	**20**	**19**
28	(10,12)	85	52.5	10.5	11.5	47.3	9.5	10.4	–	29	–	29
29	(10,13)	59	36.0	6.0	4.0	32.4	5.4	3.6	–	–	–	–
30	(11,12)	50	91.5	4.5	2.5	82.4	4.1	2.3	–	–	–	–
31	(11,13)	90	6.5	4.5	13.5	5.9	4.1	12.2	71	19	**83**	**7**

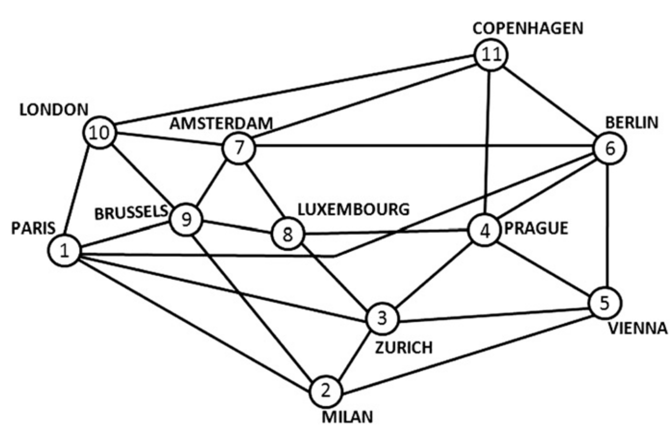

Fig. 6. Graph of Problem 3.

The fuzzy costs (modal value m and spreads α and β) of the commodities are given in Table 14 and the capacities of the arcs are given in Tables 15 and 16.

The solution of the fuzzy problem found by decomposition method and the solution of the crisp problem found by MatLab (command `linprog`) are given in Tables 15 and 16.

The decomposition method made 843 iterations in 241.30 s. The total cost was

$$\tilde{z} = (110\,717.94, 7651.12, 8054.68).$$

Multiplying the flow found by MatLab by the fuzzy cost we obtain

$$\tilde{z}_M = (110\,698.19, 7252.12, 8438.68).$$

Comparing the two fuzzy costs, $\tilde{z}_M$ and $\tilde{z}$, through the criterion of comparison it follows that

$$\tilde{z}_M = 110\,994.83 \quad \text{and} \quad \tilde{z} = 110\,818.83$$

so

$$\tilde{z}_M \underset{f}{>} \tilde{z}.$$

The decomposition method found a better solution than that obtained by MatLab considering the crisp problem.

7. Conclusions and future work

Using the fuzzy theory to treat the uncertainties in costs, it was possible to work with the problem in the fuzzy form during the resolution procedure. The results obtained proved the efficiency of the proposed method, since it found the optimal solution for all problems.

A possible future work is to apply the algorithm on large real problems. Another is to address uncertainties in the constraints (capacities) maximizing the membership functions.

Table 13
Supplies and demands—COST 239.

Node	Commodity p_1		Commodity p_2		Commodity p_3		Commodity p_4	
	Supply	Demand	Supply	Demand	Supply	Demand	Supply	Demand
1	26.7	0	0	16.9	0	19.9	57.1	0
2	18.4	0	0	2.5	0	4.6	0	2.5
3	0	3.1	0	8.3	0	4.3	0	24.7
4	0	3	0	2.4	0	4	18.4	0
5	0	3.5	0	17.4	0	1.5	0	13.3
6	0	13	36.9	0	0	12	0	3
7	0	7.4	23.8	0	15.9	0	0	0
8	0	2.5	0	2.1	0	2.5	0	2.9
9	0	11.4	0	6.1	0	0.6	0	3.9
10	13.2	0	0	20.8	38.8	0	0	20.7
11	0	14.4	15.8	0	0	5.3	0	4.5

Table 14
Fuzzy costs—COST 239.

No. arc		Cost of p_1			Cost of p_2			Cost of p_3			Cost of p_4		
(i,j)	(j,i)	m	α	β	m	α	β	m	α	β	m	α	β
1	26	210	20	20	300	10	90	270	9	81	330	11	99
2	27	280	11	9	270	95.5	4.5	243	85.95	4.05	297	105.05	4.95
3	28	490	27	6	310	13.5	3	279	12.15	2.7	341	14.85	3.3
4	29	190	10	50	255	5	25	229.5	4.5	22.5	280.5	5.5	27.5
5	30	970	30	20	950	15	10	855	13.5	9	1045	16.5	11
6	31	320	96	7	55	13	53.5	49.5	11.7	48.15	60.5	14.3	58.85
7	32	315	15	15	260	7.5	7.5	234	6.75	6.75	286	8.25	8.25
8	33	276	30	30	266	15	15	239.4	13.5	13.5	292.6	16.5	16.5
9	34	157	17	196	257	8.5	98	231.3	7.65	88.2	282.7	9.35	107.8
10	35	245	18	22	88	9	11	79.2	8.1	9.9	96.8	9.9	12.1
11	36	233	18	22	243	9	11	218.7	8.1	9.9	267.3	9.9	12.1
12	37	91	9	11	131	4.5	5.5	117.9	4.05	4.95	144.1	4.95	6.05
13	38	140	30	20	120	15	10	108	13.5	9	132	16.5	11
14	39	330	20	5	310	10	2.5	279	9	2.25	341	11	2.75
15	40	240	30	30	250	15	15	225	13.5	13.5	275	16.5	16.5
16	41	160	50	30	560	25	15	504	22.5	13.5	616	27.5	16.5
17	42	284	20	30	274	10	15	246.6	9	13.5	301.4	11	16.5
18	43	267	12	18	247	6	9	222.3	5.4	8.1	271.7	6.6	9.9
19	44	342	12	13	352	6	6.5	316.8	5.4	5.85	387.2	6.6	7.15
20	45	213	10	20	195	5	10	175.5	4.5	9	214.5	5.5	11
21	46	156	12	8	156	6	4	140.4	5.4	3.6	171.6	6.6	4.4
22	47	270	22	18	272	11	9	244.8	9.9	8.1	299.2	12.1	9.9
23	48	247	7	8	247	3.5	4	222.3	3.15	3.6	271.7	3.85	4.4
24	49	142	12	18	152	6	9	136.8	5.4	8.1	167.2	6.6	9.9
25	50	210	60	120	150	30	60	135	27	54	165	33	66

Table 15
Optimal flow – MatLab and decomposition – COST 239.

Arc			MatLab				Decomposition			
No.	(i,j)	Capacity	p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
1	(1,2)	20	–	–	–	20.00	–	–	–	17.50
2	(1,3)	20	2.50	–	–	17.50	–	–	–	20.00
3	(1,6)	40	20.70	–	–	1.50	20.70	–	–	4.00
4	(1,9)	20	3.50	–	–	16.50	6.00	–	–	14.00
5	(1,10)	30	–	–	–	1.60	–	–	–	1.60
6	(2,3)	20	2.70	1.10	1.80	14.40	5.20	1.10	1.80	11.90
7	(2,5)	10	6.00	–	–	3.10	6.00	–	–	3.10
8	(2,9)	10	9.70	–	–	–	7.20	–	–	–
9	(3,4)	2.5	–	–	–	–	–	–	–	–
10	(3,5)	10	–	–	–	7.70	–	–	–	7.70
11	(3,8)	2.5	2.10	–	–	0.40	2.10	–	–	0.40
12	(4,5)	2.5	–	–	–	2.50	–	–	–	2.50
13	(4,6)	10	–	–	–	10.00	–	–	–	10.00
14	(4,8)	2.5	–	–	–	2.50	–	–	–	2.50
15	(4,11)	2.5	–	–	–	2.50	–	–	–	2.50

Table 16
Optimal flow – MatLab and decomposition – COST 239.

Arc			MatLab				Decomposition			
No.	(i,j)	Capacity	p_1	p_2	p_3	p_4	p_1	p_2	p_3	p_4
16	(5,6)	30	–	–	–	–	–	–	–	–
17	(6,7)	20	–	–	–	5.70	–	–	–	8.20
18	(6,11)	10	7.20	–	–	2.80	7.20	–	–	2.80
19	(7,8)	2.5	–	2.10	0.40	–	–	2.10	0.40	–
20	(7,9)	10	–	10.00	–	–	–	10.00	–	–
21	(7,10)	20	–	11.70	–	8.30	–	11.70	–	8.30
22	(7,11)	2.5	2.50	–	–	–	2.50	–	–	–
23	(8,9)	2.5	–	–	–	–	–	–	–	–
24	(9,10)	10	–	–	–	10.00	–	–	–	10.00
25	(10,11)	10	4.70	–	5.30	–	4.70	–	5.30	–
26	(2,1)	20	–	–	–	–	–	–	–	–
27	(3,1)	20	–	–	–	–	–	–	–	–
28	(6,1)	40	–	16.60	–	–	–	16.60	–	–
29	(9,1)	20	–	0.30	0.90	–	–	0.30	0.90	–
30	(10,1)	30	–	–	19.00	–	–	–	19.00	–
31	(3,2)	20	–	–	–	–	–	–	–	–
32	(5,2)	10	–	–	–	–	–	–	–	–
33	(9,2)	10	–	3.60	6.40	–	–	3.60	6.40	–
34	(4,3)	2.5	–	1.60	–	0.90	–	1.60	–	0.90
35	(5,3)	10	–	5.60	2.50	–	–	5.60	2.50	–
36	(8,3)	2.5	–	–	–	–	–	–	–	–
37	(5,4)	2.5	2.50	–	–	–	2.50	–	–	–
38	(6,4)	10	0.50	1.50	4.00	–	0.50	1.50	4.00	–
39	(8,4)	2.5	–	–	–	–	–	–	–	–
40	(11,4)	2.5	–	2.50	–	–	–	2.50	–	–
41	(6,5)	30	–	23.00	4.00	–	–	23.00	4.00	–
42	(7,6)	20	–	–	20.00	–	–	–	20.00	–
43	(11,6)	10	–	4.20	–	–	–	4.20	–	–
44	(8,7)	2.5	–	–	–	–	–	–	–	–
45	(9,7)	10	1.40	–	–	2.60	1.40	–	–	0.10
46	(10,7)	20	8.50	–	4.50	–	8.50	–	4.50	–
47	(11,7)	2.5	–	–	–	–	–	–	–	–
48	(9,8)	2.5	0.40	–	2.10	–	0.40	–	2.10	–
49	(10,9)	10	–	–	10.00	–	–	–	10.00	–
50	(11,10)	10	–	9.10	–	0.80	–	9.10	–	0.80

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